

Hierarchy of N-point functions in the Λ CDM and ReBEL cosmologies

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In this work we investigate higher order statistics for the Λ CDM and ReBEL scalar-interacting dark matter models by analyzing 180 h^{-1} Mpc dark matter N-body simulation ensembles. The N-point correlation functions and the related hierarchical amplitudes, such as skewness and kurtosis, are computed using the Count-In-Cells method. Our studies demonstrate that the hierarchical amplitudes S_n of the scalar-interacting dark matter model significantly deviate from the values in the Λ CDM cosmology on scales comparable and smaller than the screening length r_s of a given scalar-interacting model. The corresponding additional forces that enhance the total attractive force exerted on dark matter particles at galaxy scales lowers the values of the hierarchical amplitudes S_n . We conclude that hypothetical additional exotic interactions in the dark matter sector should leave detectable markers in the higher-order correlation statistics of the density field. We focussed in detail on the redshift evolution of the dark matter field's skewness and kurtosis. From this investigation we find that the deviations from the canonical Λ CDM model introduced by the presence of the “fifth” force attain a maximum value at redshifts $0.5 < z < 2$. We therefore conclude that moderate redshift data are better suited for setting observational constraints on the investigated ReBEL models.

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I. INTRODUCTION

The standard hierarchical structure formation scenario assumes that the distribution of mass in the universe has grown out of primordial post-inflationary Gaussian density and velocity perturbations via gravitational instability. The resulting large-scale structures can be described in a statistical way. The two-point and higher order correlation functions are the most widely studied measures. For the standard cold dark matter paradigm – which now is a part of the commonly accepted Λ CDM model – these have been studied analytically [e.g. 1–3], as well as numerically on the basis of N-body cosmological simulations [e.g. 4–7].

Here we concentrate our study on a modified dark matter model that includes long-range scalar interactions between DM particles. We focus on the phenomenological model of such a long range “fifth” DM force proposed in a study by Farrar, Gubser and Peebles [8–13]. We follow Keselman *et al.* in dubbing this long-range scalar interaction model as ReBEL, daRK Breaking of Equivalence principle. This model was proposed as a possible remedy for some of the Λ CDM problems, which relate mostly to galaxy scales. For an excellent discussion of the motivation behind the long-range scalar-interacting model we refer to papers by Peebles [14, 15] and a recent review by Peebles & Nusser [16]. Over the past few years,

the ReBEL model has been extended and explored in a range of studies [17–23]. These studies have revealed its potential on the basis of promising results. A variety of similar models have also been studied, mostly by means of N-body simulations [24–31]. There are also additional observational arguments in favour of the “fifth force” in the dark matter sector, recently forwarded by [32].

In this paper we study the hierarchy of N-point correlation functions of the scalar-interacting DM ReBEL model. In principle, these can be used to infer observational constraints on the free parameters of the model. This is not an entirely trivial affair, since the comparison of the results with observations is somewhat complicated by a few factors: (1) galaxies do not necessarily trace the mass (biasing) and (2) in the ReBEL model the baryonic matter is insensitive to the extra scalar forces. Nonetheless, we expect that the information content of the higher order correlation functions is sufficient to distinguish between the standard DM and scalar-interacting DM paradigms.

To study the high order correlations patterns of the DM density field we use cosmological N-body simulations. The scale and resolution of the simulations are designed such that they are perfectly suited for our purpose, i.e. they address the highly nonlinear evolution at scales smaller than $\approx 10 h^{-1}$ Mpc. For the purpose of distinguishing between cosmologies these scales are particularly useful, since (1) the expected deviation of the ReBEL model from the canonical Λ CDM is maximal at these small fully nonlinear scales [17, 18], and (2) nearly all detailed observations, except for the largest galaxy catalogs, relate to the small or intermediate scales.

This paper is organized as follows: in section II we de-

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scribe scalar-interacting DM ReBEL model, followed in section III by the description of the numerical modelling. Section IV covers the issues related to the Counts-In-Cells method to sample the N-point correlation functions. The results of our study are presented in section VI, followed by the conclusions in section VII.

II. SCALAR-INTERACTING DARK MATTER MODEL

Following our previous work [18] we study the model of the ReBEL long-range scalar interactions in the dark matter sector. In this scenario dark matter particles interact by means of an additional “fifth” force mediated by a massless scalar. The extra force term is long-range, even though it is dynamically screened by a sea of light particles coupled to the scalar field [10, 11]. The resulting effective gravitational potential between two DM particles has the form [17]:

$$\Phi(\mathbf{r}) = -\frac{Gm}{r} g(x), \quad (1)$$

in which G is Newton’s constant and

$$g(x) = 1 + \beta e^{-x/r_s}. \quad (2)$$

In this expression \mathbf{r} and \mathbf{x} are the particle separation in real and comoving space. The cosmological scale factor $a(t)$ at cosmological time t is normalized to unity at the present epoch, $a(t_0) = 1$. The model is specified by means of two parameters:

- β : strength parameter
The strength parameter β is a dimensionless measure of the strength of the scalar interaction with respect to a pure Newtonian gravitational force: for $\beta = 1$ the ReBEL forces between two dark matter particles are of the same magnitude and strength as the Newtonian gravitational force.
- r_s : scale parameter
the comoving screening length in $h^{-1}\text{Mpc}$, which remains constant in the comoving frame.

The total effective force between two dark matter particles of mass m_1 and m_2 is

$$F_{DM} = -G \frac{m_1 \cdot m_2}{r^2} \left[1 + \beta \left(1 + \frac{r}{r_s} \right) e^{-\frac{r}{r_s}} \right]. \quad (3)$$

From this expression we may immediately infer that the regular Newtonian force is recovered at distances $r \gg r_s$, while for separations $r \leq r_s$ the force experienced by the dark matter particle will be enhanced or reduced with respect to the Newtonian force (depending on the sign of the strength parameter β).

III. NUMERICAL SIMULATIONS

A series of N-body numerical experiments is used to trace and investigate the growth of the large-scale structure in various cosmological scenarios. Part of the simulations concern the canonical “concordance” ΛCDM cosmology. Most simulations involve different versions of ReBEL cosmologies. In addition, 10 large-scale SCDM cosmology simulations are invoked for testing purposes.

A listing of the parameters and settings of the ensembles of the simulations is provided by Table I. Simulations of the concordance ΛCDM cosmology are labeled with LCDM, while the ReBEL ones are labeled with B and RS and related parameters indicating the β and r_s parameters of the scalar-interacting dark matter. The digits at the beginning of each label relate to the size of the simulation box. In addition to the specific scenario characteristics - such as Ω_m , Ω_Λ , Hubble parameter H , σ_8 and ReBEL Parameters β and r_s - the simulations differ in terms of the simulation box size L_{box} , number of particles N_{part} , force resolution and initial redshift z_{ini} .

The simulations in a $180 h^{-1}\text{Mpc}$ box form the core of our study, with the simulations in larger boxes kept for additional analysis. With the exception of the 256LCDMHR and 256B1RS1HR ensembles, all numerical simulations contain 256^3 dark matter particles to sample the theoretical continuum density dark matter field. Simulations 256LCDMHR and 256B1RS1HR, consisting of 512^3 dark matter particles, and simulation ensembles 180LCDMZ80 and 180B1RS1Z80 have a higher force resolution, $\varepsilon = 16.8 h^{-1}\text{kpc}$. These simulations are used to study the transients and resolution effects.

For each configuration of simulation parameters we generate an ensemble of 5-10 different simulations. This enables us to get an estimate of the cosmic variance introduced by the finite simulation box sizes. Each of the ensemble realizations is based on the same amplitude of the density field’s Fourier components, dictated by the power spectrum, while differing in terms of the corresponding random phases.

The initial density and velocity fluctuation field in all simulations are characterized by a cold dark matter spectrum. To generate the initial conditions we use the `PMcode` by Klypin & Holtzman [33], in conjunction with transfer functions computed using the `cmbfast` code by Seljak & Zaldarriaga [34]. With the exception of the Standard Cold Dark Matter SCDM model, all ΛCDM and ReBEL models start from an initial density field with a canonical ΛCDM power spectrum normalized to a linearly extrapolated density variance $\sigma_8 = 0.8$ at redshift $z = 0$ within a sphere of comoving tophat radius $R_{TH} = 8 h^{-1}\text{Mpc}$.

The 1024SCDM ensemble traces growth of structure in the Standard Cold Dark Matter (SCDM) model. Each of the 10 realizations are contained within a $1024 h^{-1}\text{Mpc}$ cubical box. Even though currently the SCDM model is very strongly disfavored by all astronomical data [e.g. 35–41], we use it as reference point and for testing purposes

TABLE I: (color on-line) Parameters used in our ensembles of simulations. No. of realizations stands for the number of different realizations of the same initial $P(k)$, β and r_s are scalar-interactions parameters. L_{box} denotes the size of the simulation box, N_{part} the number of particles and z_{ini} the initial redshift. The cosmological parameters are: Ω_m and Ω_Λ , denoting the dimensionless density parameters of the matter and cosmological constant at redshift $z = 0$, and σ_8 , the amplitude of mass fluctuations in a $8 h^{-1}\text{Mpc}$ sphere, h is the present dimensionless Hubble parameter, m_p is the particle mass, ε marks the force resolution and l denotes the mean interparticle separation.

ensemble	No. of realizations	β [$h^{-1}\text{kpc}$]	r_s [$h^{-1}\text{Mpc}$]	Ω_m	Ω_Λ	h [$100 \frac{\text{km}}{\text{s}\cdot\text{Mpc}}$]	σ_8	L_{box} [$h^{-1}\text{Mpc}$]	N_{part}	z_{ini}	m_p [$10^{10} M_\odot$]	ε [$h^{-1}\text{kpc}$]	l
1024SCDM	10	-	-	1.0	0.0	0.5	1.0	1024	256^3	35	1776.32	924	4
180LCDM	8	-	-	0.3	0.7	0.7	0.8	180	256^3	40	2.89	168	0.703
360LCDM	5	-	-	0.3	0.7	0.7	0.8	360	256^3	30	23.155	168	1.4
512LCDM	5	-	-	0.3	0.7	0.7	0.8	512	256^3	30	66.612	280	2
180B-05RS1	8	-0.5	1	0.3	0.7	0.7	0.8	180	256^3	40	2.89	168	0.703
180B02RS1	8	0.2	1	0.3	0.7	0.7	0.8	180	256^3	40	2.89	168	0.703
512B02RS1	5	0.2	1	0.3	0.7	0.7	0.8	512	256^3	30	66.612	280	2
180B1RS1	8	1	1	0.3	0.7	0.7	0.8	180	256^3	40	2.89	168	0.703
360B1RS1	5	1	1	0.3	0.7	0.7	0.8	360	256^3	30	23.155	168	1.4
512B1RS1	5	1	1	0.3	0.7	0.7	0.8	512	256^3	30	66.612	280	2
180LCDMZ80	8	-	-	0.3	0.7	0.7	0.8	180	256^3	80	2.89	16.8	0.703
180B1RS1Z80	8	1	1	0.3	0.7	0.7	0.8	180	256^3	80	2.89	16.8	0.703
256LCDMHR	10^a	-	-	0.3	0.7	0.7	0.8	256	512^3	80	1.04	16.8	0.5
256B1RS1HR	10^a	1	1	0.3	0.7	0.7	0.8	256	512^3	80	1.04	16.8	0.5

^aThese simulations have only 1 realisation, we used 10 bootstrap resamplings to obtain the estimates of the mean and standard deviation.

on the grounds that over the past decades it has been studied in great detail [e.g. 1, 4, 42–44].

To evolve the particle distribution from the initial scale factor to the present time we use the **Gadget2** Tree-Particle-Mesh code by Volker Springel [45], which we specifically modified to be able to follow the particle distribution in ReBEL force fields. The modifications allow the code to handle the long-range scalar-interacting dark matter interactions (eqn. 1-3). The detailed description of this modification may be found in our earlier work [18]. Of all simulations, we saved particle positions and velocities at redshifts $z = 5, 2, 1, 0.5$ and 0. The end product is a catalog of redshift-dependent *snapshots*.

In a simulation with a (comoving) box size of $180 h^{-1}\text{Mpc}$, a dark matter particle has a mass of $2.89 \times 10^{10} h^{-1} M_\odot$. In this case, a typical galaxy halo will contain roughly a hundred dark matter particles. This number is too small to reliably sample any relevant physical quantities of a galaxy halo. However, it is sufficient to reliably trace the non-linear evolution of the dark matter density field down to scales relevant for galaxy formation.

IV. MOMENTS OF COUNTS-IN-CELLS

Assuming the applicability of the fair-sample hypothesis [56], the volume-averaged J -point correlation function can be expressed as

$$\bar{\xi}_J = V_W^{-J} \int_S d\mathbf{x}_1 \dots d\mathbf{x}_J W(\mathbf{x}_1) \dots W(\mathbf{x}_J) \xi_J(\mathbf{x}_1, \dots, \mathbf{x}_J), \quad (4)$$

where \mathbf{x}_i is the comoving separation vector, $W(\mathbf{x})$ is a window function with volume

$$V_W = \int_S d\mathbf{x} W(\mathbf{x}), \quad (5)$$

and the integral covers the entire volume S . Because of the fair-sample hypothesis, $\bar{\xi}_J$ does not depend on the location \mathbf{x} and is a function of the window volume V_W only [1].

A. Connected Moments

There is a range of options concerning fast and accurate methods for measuring the N -point correlation

functions of a DM density field sampled by a discrete set of N particles. Our analysis is based on the moments of the distribution of counts-in-cells (hereafter CIC) [1, 42, 46, 47]. The counts define a discrete sample of the density distribution. Sampling the density field by C spherical cells, the J -th central moment of the cell counts is defined by

$$m_j(R) = \frac{1}{C} \sum_{i=1}^C (N_i - \tilde{N})^j, \quad (6)$$

where R is the comoving cell radius, N_i the number of particles found in a i -th cell and \tilde{N} the mean number of particles in cells of radius R . Following Gaztañaga[46], the connected moments μ_j of the counts may then be written as,

$$\begin{aligned} \mu_2 &= m_2, \\ \mu_3 &= m_3, \\ \mu_4 &= m_4 - 3m_2^2, \\ \mu_5 &= m_5 - 10m_3m_2, \\ \mu_6 &= m_6 - 15m_4m_2 - 10m_3^2 + 30m_2^3, \\ \mu_7 &= m_7 - 21m_5m_2 - 35m_4m_3 + 210m_3m_2^2, \\ \mu_8 &= m_8 - 28m_6m_2 - 56m_5m_3 - 35m_4^2 \\ &\quad + 420m_4m_2^2 + 560m_3^2m_2 - 630m_2^4, \\ \mu_9 &= m_9 - 36m_7m_2 - 84m_6m_3 - 126m_5m_4 + 756m_5m_2^2 \\ &\quad + 2520m_4m_3m_2 + 560m_3^3 - 7560m_3^2m_2^2. \end{aligned} \quad (7)$$

The volume-averaged correlation functions $\bar{\xi}_J$ can be computed by dividing the equations for the connected moments by \tilde{N}^J ,

$$\bar{\xi}_J = \mu_j \tilde{N}^{-J}. \quad (8)$$

B. Shot-Noise effects

Due to the discrete nature of a finite particle distribution, equation 8 is a good estimator of ξ_J only for scales where the fluid limit holds. This is satisfied if $\tilde{N} \gg 1$. For small values of \tilde{N} or, more adequately, for scales comparable with the mean inter-particle separation, the factor $\mu_j \tilde{N}^{-J}$ will be dominated by shot noise.

To correct for the shot noise effects, we use the method developed by Gaztañaga [see 46]. The method use the moment generating function of the Poisson model to calculate the net contribution by discrete noise. By including this information, one may infer expressions for the shot-noise corrected connected moments k_J :

$$\begin{aligned} k_2 &= \mu_2 - \tilde{N}, \\ k_3 &= \mu_3 - 3k_2 - \tilde{N}, \\ k_4 &= \mu_4 - 7k_2 - 6k_3 - \tilde{N}, \\ k_5 &= \mu_5 - 15k_2 - 25k_3 - 10k_4 - \tilde{N}, \\ k_6 &= \mu_6 - 31k_2 - 90k_3 - 65k_4 - 15k_5 - \tilde{N}, \\ k_7 &= \mu_7 - 63k_2 - 301k_3 - 350k_4 - 140k_5 - 21k_6 - \tilde{N}, \\ k_8 &= \mu_8 - 127k_2 - 966k_3 - 1701k_4 - 1050k_5 - 266k_6 \\ &\quad - 28k_7 - \tilde{N}, \end{aligned}$$

$$\begin{aligned} k_9 &= \mu_9 - 255k_2 - 3025k_3 - 7770k_4 - 6951k_5 - 2646k_6 \\ &\quad - 462k_7 - 36k_8 - \tilde{N}. \end{aligned} \quad (9)$$

Finally the corrected volume-averaged J -th point correlation functions of DM density field can be written as

$$\bar{\xi}_J = k_J \tilde{N}^{-J}. \quad (10)$$

We use relations described above to compute $\bar{\xi}_J$'s up to $J = 9$ from the particle distributions of our N-body cosmological simulations.

C. Sampling and errors

Because the computational cost of counting the content of cells increases with volume, we adjust the number of spherical cells used for the counts-in-cells analysis to the comoving cell radius R . We require the total number of sampling spheres to be in the range $10^5 \leq C \leq 10^6$. For the smallest scales we take $C = 10^6$, while for the largest scales the minimum number of cells is 10^5 . Within this range, the number of cells used to sample the moments, $C(R)$, scales according to

$$C(R) \propto \left(\frac{L}{R}\right)^3, \quad (11)$$

where L is the comoving simulation box width. This scaling implies the number of counted points as function of scale R to remain comparable.

Constraining the number of sampling cells is a trade-off between the requirement of keeping the sampling errors as low as possible and limits on the computational time. Because the sampling error connected with the finite number of cells C scales like C^{-1} [48], the decreasing number of cells at larger radii R leads to a corresponding growth of the intrinsic error.

In this paper we adopt the standard deviation on the mean of the J -point correlation function, determined from its estimated values ξ_J^i in the various realizations i ($i = 1, \dots, M$) within a simulation ensemble (see I),

$$\langle \bar{\xi}_J \rangle = \frac{1}{M-1} \sum_{i=1}^M \xi_J^i \quad (12)$$

as a measure for the variability and error σ_{ξ_J} in the estimate for the correlation function ξ_J ,

$$\sigma_{\xi_J} = \sqrt{\text{Var}[\bar{\xi}_J]} = \sqrt{\frac{1}{M-1} \sum_{i=1}^M (\xi_J^i - \langle \bar{\xi}_J \rangle)^2}, \quad (13)$$

The standard deviation of an ensemble obtained by averaging over its realizations concerns a conservative estimate of errors. The sampling variance is larger for different realizations within an ensemble than for measurement errors associated with the finite number of the sampling cells [42].

V. TESTING THE COUNTS-IN-CELLS METHOD

We test our implementation of the CIC method by probing its performance with respect to its estimates of the two-point correlation function ξ_2 and the three-point correlation function ξ_3 .

A. Variance and 2nd order moment

The second order moment is widely used to characterize the rms fluctuation of the matter density field on a given scale,

$$\sigma^2(R) = \bar{\xi}_2(R). \quad (14)$$

where the scale R is the comoving radius of the applied window function W .

There are two routes towards determining this factor. The first estimate of $\bar{\xi}_2(R)$ is yielded by the counts-in-cells formalism. Following the Gaztañaga formalism, CIC leads to the estimate (eqn. 10)

$$\hat{\bar{\xi}}_2[CIC](R) \equiv k_2 \tilde{N}(R)^{-2}. \quad (15)$$

where $\tilde{N}(R)$ is the number of particles in spherical cells of radius R .

A second estimate of $\sigma(R)$ is based on the power spectrum $P(k)$ of the dark matter density field in the simulations. In theory, the variance follows directly from the power spectrum of density fluctuations $P(k)$, via the integral over the comoving wave number k ,

$$\bar{\xi}_2(R) = \sigma^2(R) = \int_0^\infty \frac{dk}{2\pi^2} k^2 P(k) \hat{W}^2(kR). \quad (16)$$

With our analysis being based on counts-in-cells in spherical volumes of radius R , the natural window function is the spherical tophat function.

In the remainder, the spherical tophat function is used as window function. In Fourier space, the top-hat window function is specified by

$$\hat{W}_{TH}(kR) = 3 \frac{\sin(kR) - kR \cos(kR)}{(kR)^3}. \quad (17)$$

As a result of the discrete nature of the particles set and the finite size of the simulation box, the particle simulation cannot probe the density perturbations on scales larger than the simulation box length L and smaller than the mean particle separation,

$$l \propto N/L^3. \quad (18)$$

(for a simulation of N particles in a box of length L). For a proper comparison with the CIC inferred variance, the corresponding density field estimate integral in equation 16 is evaluated in between proper integral boundaries. The lower limit is the fundamental mode k_L , while

the Nyquist frequency k_{Nyq} represents the upper limit. For a box of size L , these are

$$k_L = \frac{2\pi}{L}, \quad k_{Nyq} = k_L \frac{N^{1/3}}{2}, \quad (19)$$

where we presume that the number of grid cells on which we have sample the initial density field is equal to the number of particles N . Hence, the power spectrum variance estimate is given by

$$\hat{\bar{\xi}}_2[Pk](R) \equiv \int_{k_L}^{k_{Nyq}} \frac{dk}{2\pi^2} k^2 P(k) \hat{W}_{TH}^2(kR). \quad (20)$$

For all simulation runs (see table I), we have computed the nonlinear power spectra directly from the resulting simulation particle distributions [57]. The integral in equation 20 is calculated from the computed nonlinear power spectra for a limited set of ensembles, those of 1024SCDM, 180LCDM, 180B-05RS1, 180B02RS1, 180B1RS1 (see table I).

1. Variance test

In Fig. 1 we present a comparison between the two estimates of the variance $\sigma^2(R)$, i.e. between the estimate $\hat{\bar{\xi}}_2[CIC]$ on the basis of the CIC method (eqn. 15) and the estimate $\hat{\bar{\xi}}_2[Pk]$ from the power spectrum integral (eqn. 16). For the ensemble of 1024SCDM simulations, we determined the variance at three different redshifts, $z = 0, 0.5$ and $z = 1$. The diagram plots the resulting variance as a function of the scale R . The symbols ($z = 0$: filled squares, $z = 0.5$: circles, $z = 1$: triangles) indicate the variance estimates on the basis of the CIC method. The continuous lines represent the variance determined from the power spectrum integral ($z = 0$: solid, $z = 0.5$: dashed, $z = 1$: dotted).

In the 1024SCDM simulation, the Nyquist frequency $k_{Nyq} \approx 0.785$ corresponds to $\sim 8 h^{-1}\text{Mpc}$. This means that the diagram in Fig. 1 suffers from a substantial level of shotnoise contribution over the range between $1 h^{-1}\text{Mpc} < R < 8 h^{-1}\text{Mpc}$. Nonetheless, the agreement between the two estimators is remarkably good down to a scale of $\approx 3 h^{-1}\text{Mpc}$, comparable to the mean inter particle separation in the 1024SCDM ensemble.

2. Variance Estimate & Model Dependence

To check whether the modified dynamics of the DM fluid in the ReBEL model affects the two variance estimators differently, we compare the resulting estimates for a range of different ReBEL models.

The top panel of fig. 2 compares the two estimates at different scales R_W for four different cosmologies: ΛCDM and a ReBEL model with strength parameter $\beta = -0.5$, a ReBEL model with $\beta = 0.2$ and one with $\beta = 1.0$. The

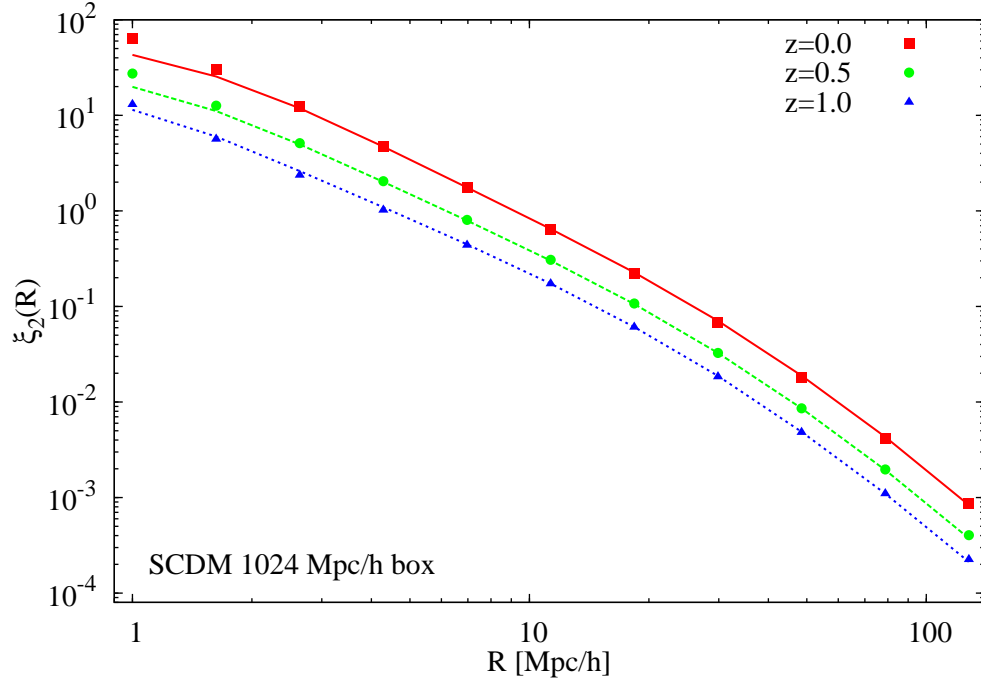


FIG. 1: (color on-line) Variance Estimators. Comparison of the CIC estimator $\hat{\xi}_2[CIC]$ and the power spectrum estimator $\hat{\xi}_2[Pk]$ of the variance as a function of scale R_W , based on the ensemble of 1024SCDM simulations. The variance is determined for three redshifts: $z = 0$, $z = 0.5$ and $z = 1.0$. The symbols represent the CIC variance estimates. Filled squares: $z = 0$, circles: $z = 0.5$, triangles: $z = 1.0$. The continuous lines indicate the power spectrum integral estimates. Solid line: $z = 0$, dashed line: $z = 0.5$, dotted line: $z = 1.0$.

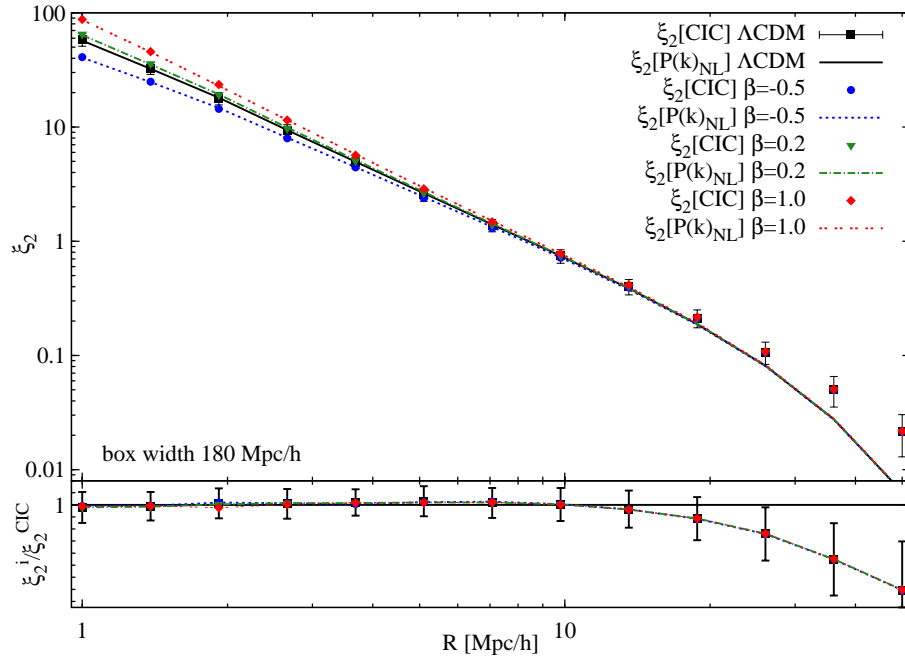


FIG. 2: (color on-line) Comparison between the CIC estimator $\hat{\xi}_2[CIC]$ and the power spectrum integral estimator $\hat{\xi}_2[Pk]$ of the variance of the density field, as a function of scale R_W . The panel shows a comparison between the two estimators for simulation ensembles of four different cosmologies. These concern the LCDM cosmology and three different ReBEL cosmologies. All simulations have a box size of $180 h^{-1} \text{Mpc}$. The power spectrum estimates are represented by continuous lines, the CIC estimates by corresponding symbols. The ensembles are: 1) 180LCDM - LCDM cosmology - solid line - square; 2) 180B-05RS1 - ReBEL cosmology with $\beta = -0.5$ - dotted line - circle; 3) 180B02RS1 - ReBEL cosmology with $\beta = 0.2$ - dot-dashed line - triangle; 4) 180B1RS1 - ReBEL cosmology - double-dotted line - diamond with $\beta = 1.0$. For clarity, we only show error bars for the 180LCDM ensemble. Top panel: regular plot of variance estimator vs. scale R_W . Bottom panel: Plot of estimator ratio $\hat{\xi}_2[Pk]/\hat{\xi}_2[CIC]$.

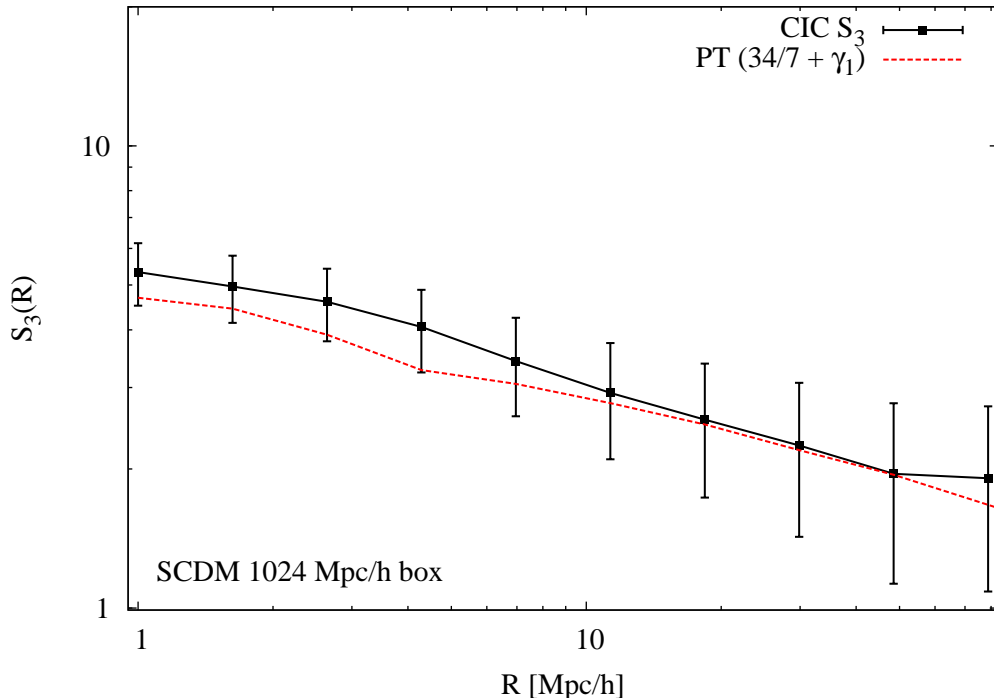


FIG. 3: (color on-line) The skewness S_3 measured for the SCDM simulation ensemble 1024SCDM. The solid line shows the CIC estimate of S_3 (eqn. 22). The dotted line shows the skewness estimated on the basis of perturbation theory (eqn. 23). The error bars (only shown for the CIC) correspond to maximal standard deviation of the ensemble (see main text for the details).

lines represent the power spectrum estimate $\hat{\xi}_{2,Pk}$ (see legend). The CIC estimates of the variance are indicated by symbols of the same colour as the lines, listed in the legenda. The difference between the two estimates may be best appreciated from the bottom panel, which shows the ratio between the two estimators, $\hat{\xi}_2[Pk]/\hat{\xi}_2[CIC]$.

Both panels clearly shows that for all four different cosmologies the two estimators agree very well for scales ranging from $\approx 10 h^{-1}\text{Mpc}$ down to the smallest scales that we analyzed, $\approx 1 h^{-1}\text{Mpc}$. On larger scales, from $\approx 20 h^{-1}\text{Mpc}$ (roughly 1/10th of the box width), we see a marked disagreement between the two estimators. This difference rapidly increases towards larger scales, with the CIC estimate systematically increasing as a function of scale with respect to the power spectrum value. Nonetheless, the fact that the difference between the two estimates is identical for the different model ensembles, in terms of character and scale at which they start to diverge, indicates that the accuracy achieved by the CIC method is the same for each of the cosmologies.

B. Third order moment: the S_3 test

The second order density field statistic, represented by the two-point correlation function, is not sufficient for characterizing the density field beyond the linear phase of structure evolution. Moving into the quasi-linear phase, we start to discern the gravitational contraction of over-

dense regions into sheetlike and filamentary patterns and compact dense haloes and the volume expansion of low density void regions. To be able to follow and characterize this process, we need to turn to the higher order moments of the density field.

To test the performance of the CIC estimator, we turn to the reduced third moment of the density field. The skewness S_3 is defined as

$$S_3 \equiv \frac{\bar{\xi}_3}{\bar{\xi}_2^2} = \frac{\bar{\xi}_3}{\sigma^4} \quad (21)$$

An estimate of S_3 can therefore be readily obtained on the basis of the corrected volume-averaged 3-point correlation function of the dark matter density field (see eqn. 10 and eqn. 9),

$$\hat{S}_3[CIC](R) = \frac{\bar{\xi}_3}{\sigma^4} = \frac{k_3 \tilde{N}(R)^{-3}}{k_2^2 \tilde{N}(R)^{-4}} = \frac{k_3}{k_2^2} \tilde{N}(R), \quad (22)$$

where $\tilde{N}(R)$ is the number of particles in spherical cells of radius R .

An alternative estimate of the skewness finds its origin in weakly nonlinear perturbation theory (PT, [1, 2, 47, 49]). Juszkievicz *et al.*[2] showed that a good approximation for the skewness S_3 of the field, smoothed with the spherical top-hat window, is given by

$$S_3 = \frac{34}{7} + \gamma_1, \quad (23)$$

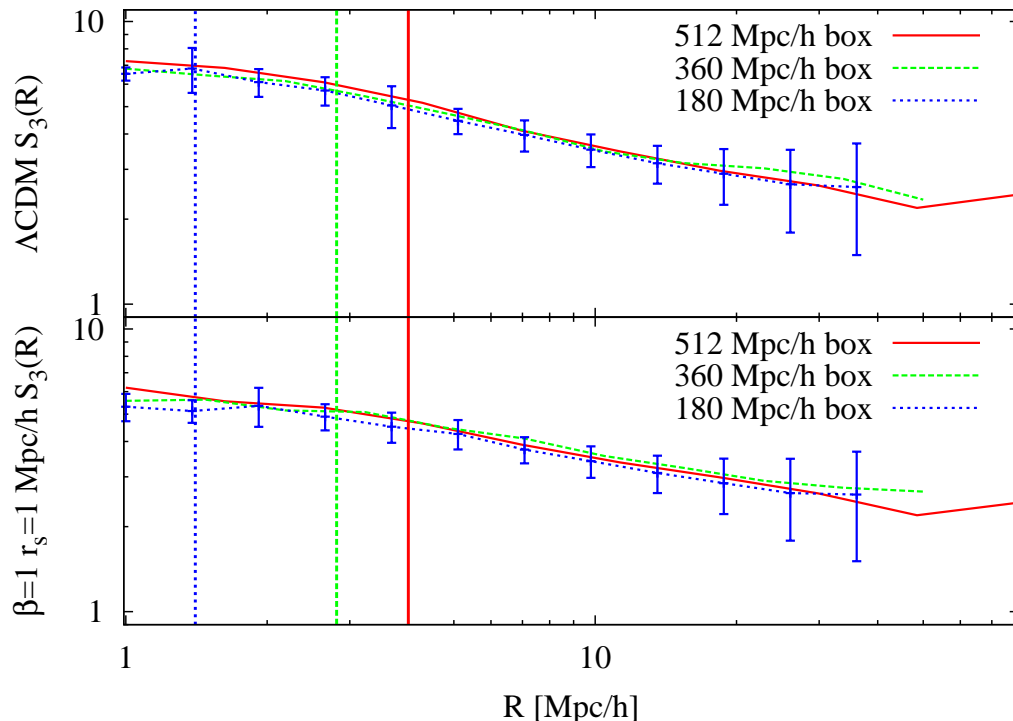


FIG. 4: (color on-line) Comparison of the measured skewness $S_3(R)$ in ensembles with different box sizes. Top panel: results for pure Λ CDM model simulations. Bottom panel: results for the ensembles for a ReBEL model with scalar interaction parameters $\beta = 1$ and $r_s = 1 \, h^{-1} \text{Mpc}$. In each panel we plot the $S_3(R)$ relation for three different ensembles of the same model, one in a box with a width of $180 \, h^{-1} \text{Mpc}$ (blue dotted line), of $360 \, h^{-1} \text{Mpc}$ (green dashed line) and one of $512 \, h^{-1} \text{Mpc}$ (red solid line). The vertical lines show the corresponding values of the Nyquist scale $R_{Nyq} = 2\pi/k_{Nyq} \sim 2l$ for each of these simulation boxes. The error bars represent the 1σ errors in the 180LCDM (top panel) and 180B1RS1 (bottom panel) ensembles.

where γ_1 is the logarithmic slope of the variance, defined as

$$\gamma_1 = -(n+3) = \frac{d \log \sigma^2(R)}{d \log R}, \quad (24)$$

where n is the slope of the power spectrum at scale R . The term $34/7$ is a well-known result pertaining to the unsmoothed field (see [1]). For the estimate of the skewness $\hat{S}_3[PT]$ based on this result, we use the estimate of the variance $\sigma^2(R)$ obtained via the integral over the non-linear power spectrum $P(k)$ for a tophat filter $W(kR)$, ie. from $\hat{\xi}_2[Pk](R)$ (eqn. 20),

$$\hat{S}_3[PT](R) = \frac{34}{7} + \frac{d \log \hat{\xi}_2[Pk](R)}{d \log R}. \quad (25)$$

In fig. 3 we have compared the two estimates of the skewness S_3 for the SCDM simulations in the 1024SCDM ensemble, over a range of $1 \, h^{-1} \text{Mpc} < R < 80 \, h^{-1} \text{Mpc}$. The solid line represents the skewness measured directly on the basis of the counts in cells method (eqn. 22), while the dotted line is the perturbation theory prediction (eqn. 25). The error bars, here shown for the CIC estimates, are the maximal standard deviation of the measurements for the simulations in the 1024SCDM ensemble

(taking into account that at each different scale R we use a *different* number of sampling cells).

Overall, we find that the two skewness estimators are in reasonable agreement with each other, in particular on linear and mildly nonlinear scales, $8 \, h^{-1} \text{Mpc} < R < 80 \, h^{-1} \text{Mpc}$, exactly as expected and reported by many other authors [2, 4, 42].

1. The box size test

The effects of finite volume on the statistics of large scale structure have been extensively studied in several studies [50]. For most of the results presented in this paper, finite volume effects are rather unimportant. We focus mainly on a direct comparison between observables of the canonical Λ CDM model and those of the scalar-interacting dark matter ReBEL models. As long as any of the finite volume induced artefacts affects each of the cosmological models to a comparable extent, we need not worry about their influence on the results of our study.

Nonetheless, there is one factor which needs to be investigated in some detail. The new physics of the dark sector scalar-interacting ReBEL models involves a new fundamental and intrinsic scale, the screening length r_s .

It is a priori unclear in how far the relation between the length L of the simulation box and the screening length r_s of the ReBEL model will be of influence on the counts-in-cell measurement of various moments.

To evaluate whether the finite box size has any impact on the measured values of S_3 , we have run a set of simulations for two different models. One model is the Λ CDM model, whose gravitational force law is entirely scale-free, while the other model is a ReBEL model characterized by an intrinsic force scale. We chose a ReBEL model with strength factor $\beta = 1$ and scale parameter $r_s = 1 h^{-1}\text{Mpc}$. Each of the two sets of simulations contain three ensembles of the same cosmological model. The first ensemble has a $180 h^{-1}\text{Mpc}$ simulation box, the second a $360 h^{-1}\text{Mpc}$ box and the third one a $512 h^{-1}\text{Mpc}$ box.

In figure 4 we follow the trend of the skewness $S_3(R)$ as a function of scale R , for each of the simulation ensembles. The top panel shows the results for the three sets of Λ CDM simulations, the 180LCDM simulations in a $180 h^{-1}\text{Mpc}$ box (dotted line), the 360LCDM simulations in a $360 h^{-1}\text{Mpc}$ box (dashed line) and the 512LCDM simulations in a $512 h^{-1}\text{Mpc}$ box (solid line). The same is repeated for the ReBEL model in the bottom panel, with the 180B1RS1 simulations in a $180 h^{-1}\text{Mpc}$ box (dotted line), the 360B1RS1 simulations in a $360 h^{-1}\text{Mpc}$ box (dashed line) and the 512LCDM simulations in a $512 h^{-1}\text{Mpc}$ box (solid line). In the figure we have also indicated the location of the Nyquist scale, $R_{Nyq} \equiv 2\pi/k_{Nyq}$, of each of the three simulation boxes. The three vertical lines mark their position.

We find that in the Λ CDM case, the measured skewness in the simulation ensembles with different box size agree very well over the entire range we probed, from the largest measured scales $\sim 30 h^{-1}\text{Mpc} < R < 80 h^{-1}\text{Mpc}$ down to the smallest scales of $1 h^{-1}\text{Mpc} < R < 8 h^{-1}\text{Mpc}$. Interestingly, we also find a similar good agreement between the simulation ensembles of the ReBEL model. Moreover, we also find a surprisingly good agreement at scales where we expect two-body effects to start to dominate, below the Nyquist scale of the simulation.

Given the fact that the measured S_3 values remain consistent over such a wide range of scales and seems independent of the size of the simulation box size, we conclude that the effect of a different ratio r_s/L of intrinsic force scale to box size has negligible, if any, effect on the measurement of statistical moments.

2. Transients

The `PMcode` that we use to generate the initial conditions is based on the Zeldovich Approximation (ZA) method [51]. It is well known that the Zeldovich approximation introduces an artificial level of skewness and additional higher order hierarchy moments into the density field [52, 53]. A sufficient number of simulation time-

steps is required for the true particle dynamics to take over and to relax these transient artifacts. An alternative approach is to resort to second order Lagrangian perturbation theory schemes for setting up the initial conditions of simulations [52–55].

Because of the above, the initial redshift of a cosmological simulation is an important factor in determining the statistical reliability of the cosmological numerical experiment. In general, for the purpose of comparing density fields and cumulants in different models we need to be less concerned about the net amplitude of the transients as they will have the same magnitude in all models.

Nonetheless, there is an additional factor that depends on the initial redshift and which only affects the ReBEL models. The intrinsic scalar force of these models should be able to act as long as possible, in order to account for an optimal representation of their impact on the dark matter density field. If the ReBEL simulations are evolved too far by means of the Zeldovich approximation and their dynamical evolution started too late, the deviation of the ReBEL dark matter density field from the one in the Λ CDM simulations will diminish.

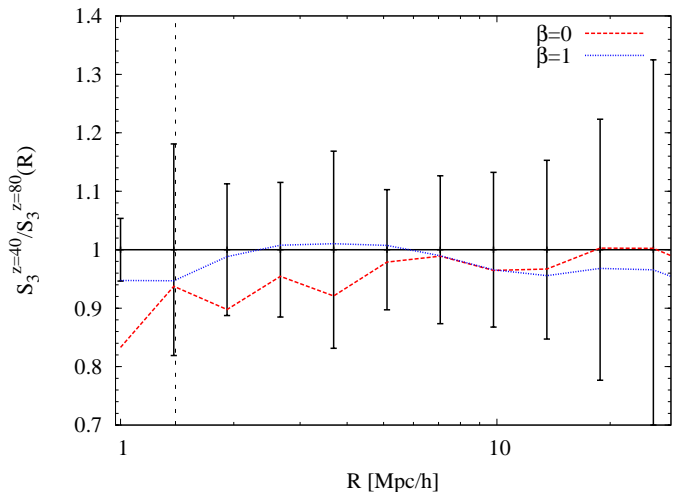


FIG. 5: (color on-line) Test of transients effects for $180 h^{-1}\text{Mpc}$ ensembles. $S_3(z_i = 40)/S_3(z_i = 80)$: ratio of skewness $S_3(z_i = 40)$ measured for the simulation ensemble started at redshift $z_i = 40$ to the skewness of the simulations started at redshift $z_i = 80$. The ratio $S_3(z_i = 40)/S_3(z_i = 80)$ is plotted as a function of scale R . The solid black horizontal line represents the unity line for which $S_3(z_i = 40) = S_3(z_i = 80)$. The dashed vertical line marks the Nyquist scale for the simulations in a $180 h^{-1}\text{Mpc}$ box, at $2\pi/k_{Nyq} = 1.38 h^{-1}\text{Mpc}$. The ratio $S_3(z_i = 40)/S_3(z_i = 80)$ is plotted for two different situations. Red dashed line: skewness ratio for the two ensembles of Λ CDM simulations, 180LCDM and 180LCDMZ80. Blue dotted line: skewness ratio for the two ReBEL simulation ensembles with $\beta = 1$ and $r_s = 1 h^{-1}\text{Mpc}$, 180B1RS1 and 180B1RS1Z80. The error bars represent the 1σ values determined for the 180LCDM ensemble. The errors in the other ensembles have a similar magnitude.

In order to quantify the possible effects of the transients, we have performed a series of auxiliary simulation ensembles. These contain 256^3 DM particles placed in boxes of the box width $180 h^{-1}\text{Mpc}$ and have 10 times better force resolution. There are two ensembles, one for the ΛCDM model, 180LCDMZ80, and one for the ReBEL model with $\beta = 1$ and $r_s = 1 h^{-1}\text{Mpc}$, 180B1RS1Z80. We will compare them with our main ensembles for the same models, 180LCDM and 180B1RS1. Therefore the ensembles of each model will differ only in the force resolution and the redshift at which the N-body calculation is started, one at $z_i = 80$ and the other at $z_i = 40$.

The results for the direct comparison of the skewness S_3 in the $z_i = 80$ models and the $z_i = 40$ models, in terms of their ratio $S_3^{z_i=40}/S_3^{z_i=80}$, are plotted in figure 5. The blue dotted line represents the ratio for the ReBEL ensemble, the red dashed line for the ΛCDM ensemble (for which $\beta = 0$). For reference, the black horizontal solid line indicates the unity ratio $S_3^{z_i=40}/S_3^{z_i=80} = 1$, while the vertical line marks the Nyquist scale $2\pi/k_{\text{Nyq}} \cong 1.4 h^{-1}\text{Mpc}$ for these simulations. The error bars mark 1σ errors in the 180LCDM ensemble, with errors in the other three ensembles being of the same order.

We note that the visible transients effects are, if real, very small. The skewness ratio curves lie very close to the unity line $S_3^{z_i=40}/S_3^{z_i=80} = 1$. Their deviations from unity are smaller than the 1σ errors, with discrepancies not exceeding the 10% level. On the basis of this we may conclude that the redshift of the initial conditions of our main ensembles, at $z_i = 40$, is sufficiently high to assure that any effects of possible transient are negligible for our analysis.

3. Resolution

The last important effect we must check is the impact of the mass and force resolution used in our simulations on the measured quantities. The mass resolution is related to the mean inter-particle separation, while the force resolution corresponds to the scale at which the force prescription of the simulation code exactly recovers the intended Newtonian - or ReBEL - force.

To investigate the impact of these resolution factors on the measurement skewness we use the high force resolution ensembles 180LCDMZ80 and 180B1RS1Z80, as well as two single high mass and force resolution runs, 256LCDHR and 256B1RS1HR (see table I). For these two simulations we use bootstrap resampling to obtain averages of mean and variance of the measured moments. This is accomplished as follows. We randomly cast a large number of spherical cells over the entire simulation volume. This ranges from 2×10^8 cells with $R = 1 h^{-1}\text{Mpc}$ to 2×10^6 cells for $R = 30 h^{-1}\text{Mpc}$. Ten sets of measurements were constructed, each consisting of a random subset of 10% of the casted spheres.

We may assess the resolution effects on the basis of the plot in figure 6. It depicts the ratio of the skewness in

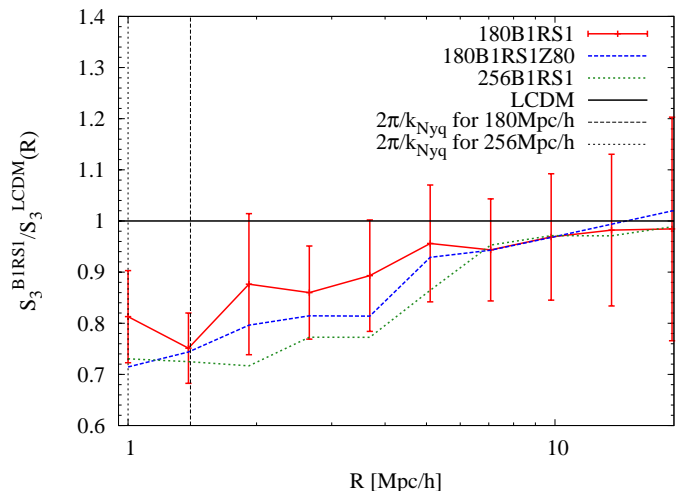


FIG. 6: (color-on-line) Test of resolution effects on computed skewness S_3 . Plotted are the ratios S_3^{B1RS1}/S_3^{LCDM} of the skewness obtained in the ReBEL model with $\beta = 1$ and $r_s = 1 h^{-1}\text{Mpc}$ to the skewness measured in the LCDM model, for three simulations with different force and mass resolutions. The lines depict relevant ratios for lower resolution run 180B1RS1 (solid line), high force resolution run 180B1RS1Z80 (dashed line) and high mass and force resolution run 256B1RS1HR (dotted line). The vertical dashed line marks Nyquist scale for runs with 256^3 particles, while the vertical dotted line depicts the same scale for 512^3 particles. The error bars represent the 1σ values determined for the 180B1RS1 ensemble.

three different ReBEL ensembles to that of the skewness in the LCDM model, S_3^{B1RS1}/S_3^{LCDM} . Each of the three ReBEL models have the same ReBEL parameters, $\beta = 1$ and $r_s = 1 h^{-1}\text{Mpc}$, but differ in resolution. The lower resolution run is 180B1RS1 (solid line), the high force resolution run is 180B1RS1Z80 (dashed line), while the high mass plus high force resolution run is that of 256B1RS1HR (dotted line). The error-bars marking the skewness ratio of the 180B1RS1 run are the 1σ errors for 180B1RS1 ensemble.

Even though we find that the simulations with a higher resolution show a systematically higher signal level at scales $R < 7 h^{-1}\text{Mpc}$, this effect is entirely contained within - or at best marginally above - the 1σ errors of the 180LCDM ensemble. We may therefore conclude that an increase in the force and/or mass resolution of the simulation does not yield a significant improvement of the signal level. This reassures us that the simulation ensembles used in our main study yield good and reliable estimates of the quantities which we study.

C. CIC test summary

In all, we may conclude from the various tests of the Count-in-Cell method that it is perfectly suited for studying the impact of long-range scalar interactions on

the higher-order correlation statistics of the dark matter density field.

VI. MOMENT ANALYSIS OF N-BODY ENSEMBLES

Having ascertained ourselves of the reliability of the CIC machinery, we will present and discuss the results of the correlation function analysis of our N-body experiments. The intention of this study is the identification of discriminative differences between the canonical Λ CDM cosmology and a range of scalar interaction ReBEL models.

We address two aspects of the resulting dark matter distributions. The first concerns a complete census of the hierarchy amplitudes S_n , from $n = 3$ to $n = 8$ for a set of three different ReBEL model simulations and a similar ensemble of Λ CDM simulations. In addition, in order to assess the redshift evolution of these statistical measures, we focus on the redshift dependence of the skewness and kurtosis.

A. Hierarchy amplitudes

The hierarchy amplitudes $S_n(R)$ of order n are conventionally defined as,

$$S_n(R) = \frac{\bar{\xi}_n}{\bar{\xi}_2^{n-1}} = \bar{\xi}_n \sigma^{-2(n-1)}, \quad (26)$$

with the volume-averaged correlation functions $\bar{\xi}_n(R)$ and variance $\sigma^2(R)$ implicitly depending on the scale R .

1. General Trends

In figure 7 and 8 we plot the measured S_n 's, from $n = 3$ up to $n = 8$, for all simulation ensembles with boxwidth $180 h^{-1}\text{Mpc}$ (see table I). These two figures represent the key result of this study.

The volume-averaged N-point correlation functions $\bar{\xi}_n$ have been computed by means of the CIC method, following the description in section IV. The simulations for which the hierarchy amplitudes have been computed are the 180LCDM set of Λ CDM simulations and the 180B-05RS1, 180B02RS1 and 180B1RS1 simulations of the ReBEL models with scalar interaction scale parameter $r_s = 1 h^{-1}\text{Mpc}$ and strength parameter $\beta = -0.5$, $\beta = 0.2$ and $\beta = 1.0$. The $\beta = -0.5$ case, whose physical effect is that of a repulsive scalar ReBEL force, does not have a real physical motivation. It is mainly included for reference, in order to outline the impact of the β strength parameter on the final nonlinear density field. For all model simulations we have calculated the hierarchy amplitudes S_n at 12 logarithmically spaced scales within

the range of $1 h^{-1}\text{Mpc} < R < 36.09 h^{-1}\text{Mpc}$. The exact values of these 12 scale values R are listed in table II.

Figures 7 and 8 plot the hierarchy amplitudes S_n as a function of scale R . The two figures are complementary: in figure 8 the S_n are shown separately for each of the cosmological models, while figure 7 superimposes the curves for each of the models in order to highlight their differences. In addition, to provide an impression of the relative differences between hierarchy amplitudes in each of the cosmological models, figure 9 plots the ratio between the $S_n(R)$ between each ReBEL model and the concordance Λ CDM models. In figure 7, each cosmological model is indicated by a different line types. The canonical Λ CDM model is indicated by the black solid line, the ReBEL model with $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ by the blue dotted line, the ReBEL model with $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ by the green dot-dashed line and the ReBEL model with $\beta = 1.0$, $r_s = 1 h^{-1}\text{Mpc}$ by the red dashed line. Figure 8 also includes the error bars of the measured S_n values, restricted to their upper half for purposes of clarity. For reference, we have listed the values of the standard deviation for the skewness S_3 and kurtosis S_4 in tables II and III. The thin dashed vertical lines in figures 7 and 8 mark the Nyquist scale $R_{Nyq} = 2\pi/k_{Nyq} \approx 1.4 h^{-1}\text{Mpc}$ for the Λ CDM and ReBEL simulations, which for these realizations is double the mean inter-particle separation $2l$. We consider the computed quantities on scales below the Nyquist scale as unreliable, and exclude them from further analysis in this study.

There are some clear trends in the behaviour of the S_n hierarchy. At large scales, $R > 10 h^{-1}\text{Mpc}$, all cosmologies agree on the S_n . This is straightforward to understand because at these large scales the ReBEL models are practically equivalent to the Λ CDM cosmology. The differences between the models become distinct at scales $R \leq 10 h^{-1}\text{Mpc}$, where the effect of the scalar ReBEL force kicks in. We discern a systematic trend, with all S_n consistently higher than the Λ CDM values for the ReBEL model with $\beta = -0.5$, consistently lower than the Λ CDM values for the ReBEL model with $\beta = 1.0$ and the values for the ReBEL model with $\beta = 0.2$ straddling tightly around the Λ CDM values. We also notice that the differences between the models increase systematically as a function of order n (see fig. 9). This may be easily understood from the higher sensitivity of the higher moments to the changing shape of the density probability function, and hence to the changes in the dark matter density distribution.

2. Skewness and Kurtosis

In the observational reality, beset by various sources of noise, it may be cumbersome to get reliable estimates of higher order moments. On the other hand, we may expect reasonably accurate estimates of the third and fourth order moments, the skewness and kurtosis. The

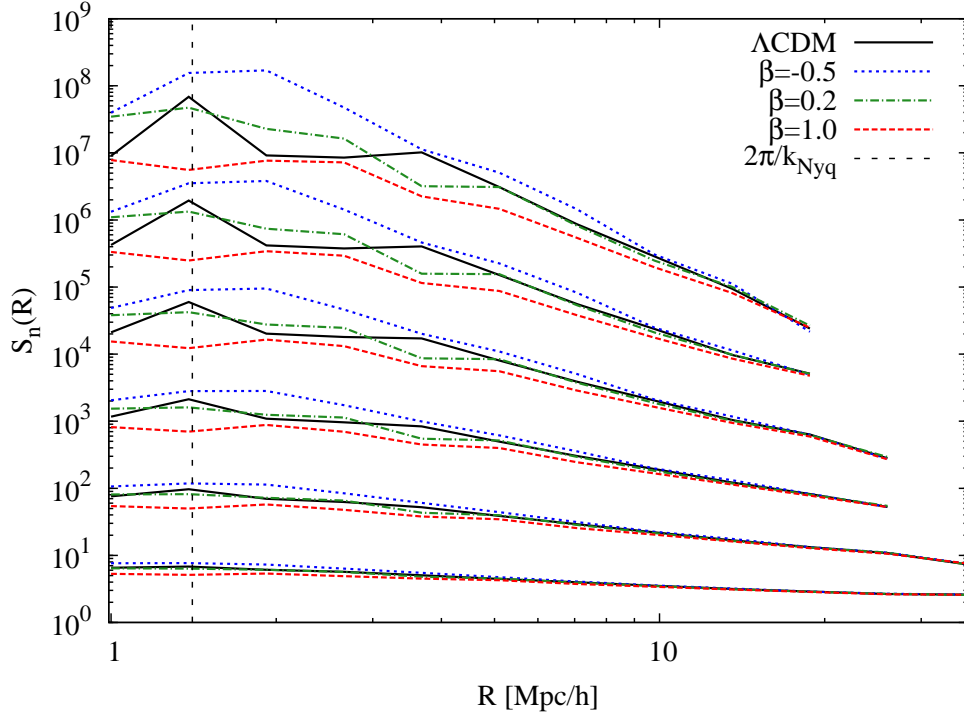


FIG. 7: (color on-line) Hierarchical amplitude $S_n(R)$ as a function of scale R . Plotted are S_n , for $n = 3$ to $n = 8$, for four different simulation ensembles: the canonical Λ CDM model 180LCDM simulations (solid black line), the $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1 (blue dotted line), the $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1 (green dot-dashed line) and the $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1 (red dashed line). The S_3 curves have the lowest amplitude, with the amplitude of the S_n curves systematically increasing as a function of n . The thin vertical dashed line marks the Nyquist scale $R_{Nyq} \approx 1.4 h^{-1}\text{Mpc}$.

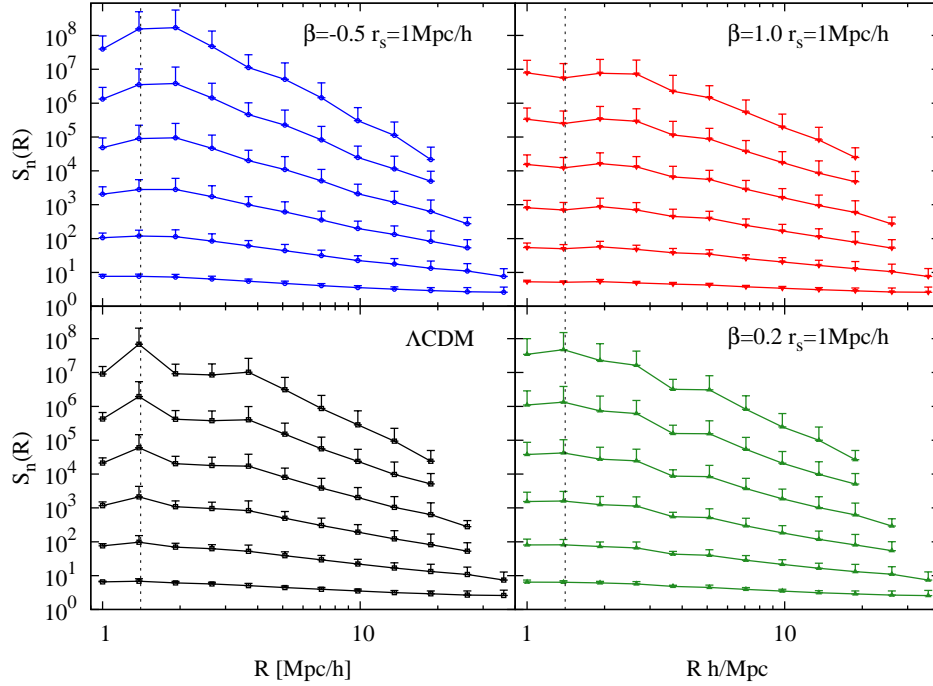


FIG. 8: (color on-line) Hierarchical amplitudes $S_n(R)$ as a function of scale R . Equivalent to fig. 7, the S_n are plotted in a separate panel for each cosmological model. Top left: $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1; top right: $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1; bottom left: the canonical Λ CDM model 180LCDM simulations; bottom right: $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1. The S_3 curves have the lowest amplitude, with the amplitude of the S_n curves systematically increasing as a function of n . The thin vertical dashed line marks the Nyquist scale $R_{Nyq} \approx 1.4 h^{-1}\text{Mpc}$.

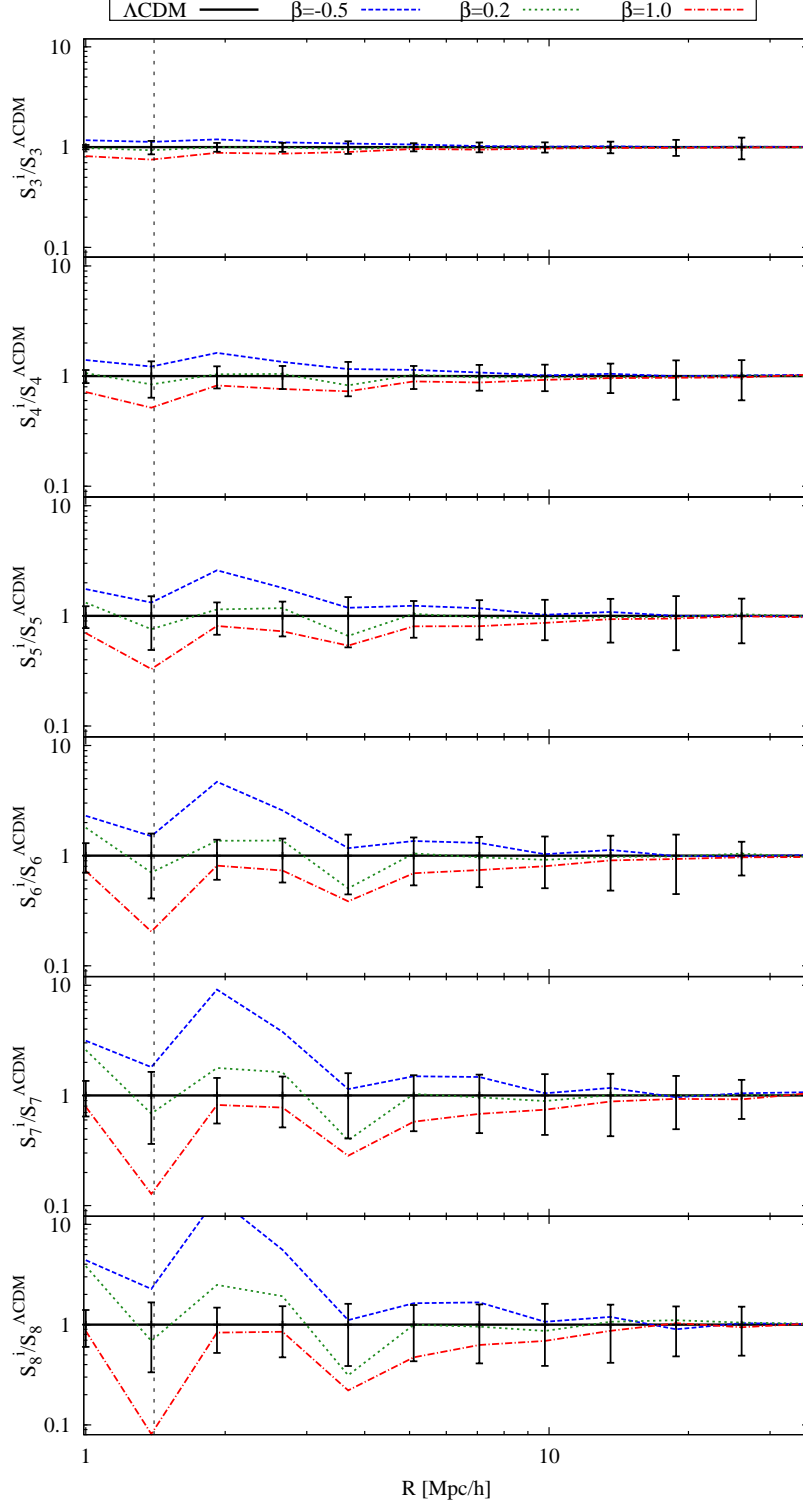


FIG. 9: (color on-line) Hierarchical amplitude $S_n(R)$ as a function of scale R : ratio $S_n(R)/S_n^{\Lambda\text{CDM}}(R)$ of the hierarchy amplitudes in any of the simulation ensembles to the hierarchy amplitudes in the concordance ΛCDM cosmology simulations. From top to bottom panel: $S_3/S_3^{\Lambda\text{CDM}}$ to $S_8/S_8^{\Lambda\text{CDM}}$. In each panel we plot the curves for four different simulations ensembles: the canonical ΛCDM model 180LCDM simulations, by definition equal to unity (solid black line), the $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1 (blue dotted line), the $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1 (green dot-dashed line) and the $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1 (red dashed line). The S_3 curves have the lowest amplitude, with the amplitude of the S_n curves systematically increasing as a function of n . The thin vertical dashed line marks the Nyquist scale $R_{Nyq} \approx 1.4 h^{-1}\text{Mpc}$. The error-bars correspond to 1σ scatter of the 180LCDM ensemble.

TABLE II: Measured values of the S_3 hierarchy amplitude at redshift $z = 0$, for four simulation ensembles in different cosmologies. The models are the Λ CDM model 180LCDM simulation, the $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1, the $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1 and the $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1.

R ($h^{-1}\text{Mpc}$)	180LCDM	180B-05RS1	180B02RS1	180B1RS1
01.00	6.53 ± 0.35	7.65 ± 1.08	6.40 ± 1.00	5.30 ± 0.59
01.38	6.82 ± 1.23	7.69 ± 1.23	6.35 ± 0.74	5.13 ± 0.47
01.92	6.10 ± 0.69	7.29 ± 1.45	6.09 ± 0.70	5.35 ± 0.84
02.66	5.70 ± 0.66	6.35 ± 1.37	5.66 ± 1.00	4.90 ± 0.52
03.68	5.05 ± 0.85	5.48 ± 0.87	4.80 ± 0.39	4.51 ± 0.55
05.10	4.45 ± 0.46	4.72 ± 0.81	4.49 ± 0.73	4.26 ± 0.51
07.07	3.97 ± 0.50	4.07 ± 0.61	3.90 ± 0.51	3.74 ± 0.40
09.80	3.52 ± 0.46	3.56 ± 0.50	3.50 ± 0.47	3.41 ± 0.43
13.57	3.15 ± 0.48	3.21 ± 0.54	3.12 ± 0.47	3.10 ± 0.47
18.80	2.89 ± 0.65	2.90 ± 0.66	2.87 ± 0.65	2.85 ± 0.63
26.05	2.66 ± 0.86	2.68 ± 0.87	2.66 ± 0.87	2.63 ± 0.84
36.09	2.60 ± 1.10	2.60 ± 1.10	2.57 ± 1.08	2.60 ± 1.09

TABLE III: (color on-line) Measured values of the S_4 hierarchy amplitude at redshift $z = 0$, for four simulation ensembles in different cosmologies. The models are the Λ CDM model 180LCDM simulation, the $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1, the $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1 and the $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1.

R ($h^{-1}\text{Mpc}$)	180LCDM	180B-05RS1	180B02RS1	180B1RS1
01.00	75 ± 12	106 ± 40	81 ± 39	54 ± 19
01.38	97 ± 55	118 ± 58	82 ± 35	50 ± 16
01.92	70 ± 20	113 ± 68	72 ± 26	57 ± 25
02.66	63 ± 20	84 ± 54	66 ± 33	48 ± 16
03.68	52 ± 28	61 ± 27	43 ± 9	38 ± 13
05.10	39 ± 12	44 ± 23	40 ± 19	35 ± 12
07.07	29 ± 10	31 ± 14	28 ± 11	26 ± 8
09.80	22 ± 8	22 ± 9	22 ± 8	20 ± 7
13.57	17 ± 7	18 ± 8	16 ± 7	16 ± 7
18.80	13 ± 8	13 ± 9	13 ± 8	13 ± 8
26.05	11 ± 7	11 ± 7	11 ± 7	11 ± 7
36.09	7 ± 5	8 ± 5	7 ± 5	8 ± 5

question is whether the presence or absence of ReBEL scalar forces may be deduced from the behaviour of these moments. To evaluate the discriminatory powers of S_3 and S_4 we list the measured values of these hierarchy amplitudes in tables II and III.

Assessing the data presented in these tables reveals that S_3 and S_4 values converge to within 1σ around the Λ CDM values for scales larger than $R = 9.8 h^{-1}\text{Mpc}$. As we turn towards smaller scales R , the ReBEL model values for the skewness and kurtosis display an increasingly large difference with respect to the Λ CDM value. In other words, at these small (mildly) nonlinear scales we observe a direct imprint of the scalar forces on the density field moments.

At scales comparable to the screening length, ReBEL models with a positive strength parameter β have a lower skewness and kurtosis value than those for the canonical Λ CDM model. The difference is smaller for ReBEL models with a lower β , and turns into a higher value as β turns negative. Seen as a function of scale, the difference decreases towards larger scales R .

For the $\beta = 1.0$ 180B1RS1 simulations the value of S_3 at $R \sim 1.4 h^{-1}\text{Mpc}$ is $\approx 25\%$ lower than the value for the Λ CDM model, while the discrepancy is in only the order of $\approx 10\%$ at $R = 3.68 h^{-1}\text{Mpc}$ and has dropped towards $\leq 5\%$ for $R \geq 7 h^{-1}\text{Mpc}$. The differences are more prominent in the case of the kurtosis S_4 . For $R \sim 1.4 h^{-1}\text{Mpc}$ the value of S_4 is smaller than the Λ CDM value by no less than $\approx 48\%$, decreasing towards $\approx 27\%$ at $R = 3.68$ and to less than 10% at $R \geq 7 h^{-1}\text{Mpc}$.

The differences between the cosmological models are therefore less substantial for the skewness S_3 than for the kurtosis S_4 . On the condition that it is possible to obtain reliable estimates for S_4 in the observational reality, this leads us to the conclusion that the kurtosis may be better suited as tracer of ReBEL signatures in the density field. More detailed studies and simulations, including baryons and mock galaxy samples, will be necessary to make a final choice for the optimal marker of ReBEL cosmology in observational catalogues.

B. Redshift evolution

In our previous study [18] we found that the amplitude of the deviation of the two-point correlation function ξ of the ReBEL model to that of the Λ CDM model changes with redshift. This suggests a similar evolution of higher order moments like S_3 and S_4 , prodding us to assess the redshift evolution of skewness and kurtosis.

To this end, we study the archive of five snapshots – at redshifts $z = 5., 2., 1., 0.5$ and $z = 0$ – which for each simulation in the four $180 h^{-1}\text{Mpc}$ ensembles were saved: 180LCDM, 180B-05RS1, 180B02RS1, and 180B1RS1.

The redshift evolution of the skewness and kurtosis in the four simulation ensembles can be followed in figure 11. In the lefthand column we plot the ratio $S_3^{\text{ReBEL}}/S_3^{\Lambda\text{CDM}}$ of the skewness in the three different ReBEL models to the one for the canonical Λ CDM model in a sequence of five panels, for the five subsequent redshift snapshots which we analyzed, from $z = 5.$ (bottom) to $z = 0$ (top). The righthand column is organized in an equivalent manner for the kurtosis ra-

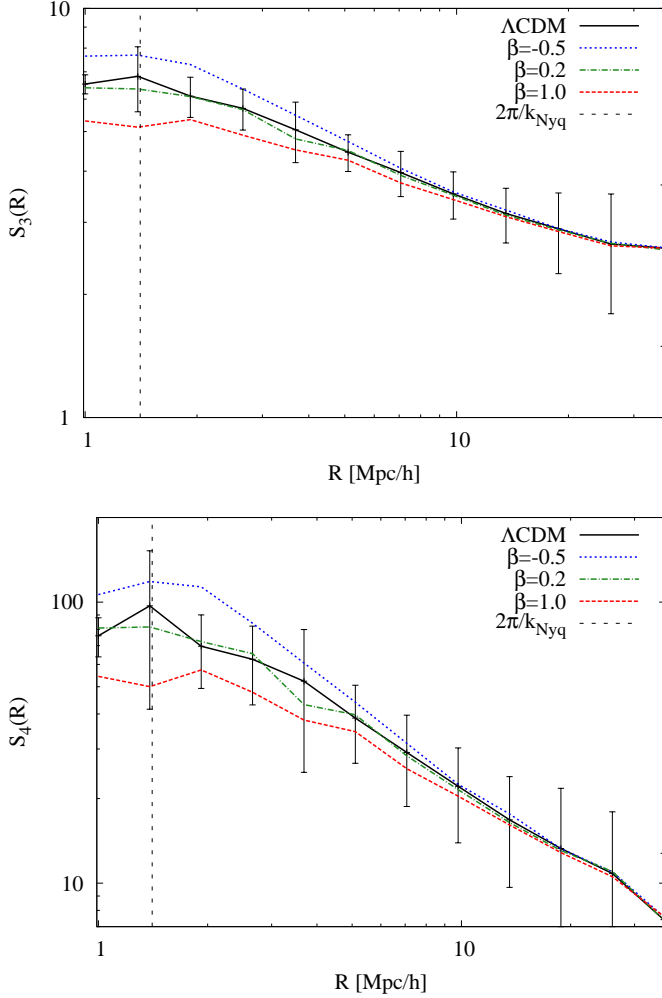


FIG. 10: (color on-line) Skewness $S_3(R)$ and kurtosis $S_4(R)$ as a function of scale R . Plotted are $S_3(R)$ and $S_4(R)$ in four $180 h^{-1}\text{Mpc}$ box simulation ensembles: the canonical ΛCDM model 180LCDM simulations (solid black line), the $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B-05RS1 (blue dotted line), the $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B02RS1 (green dot-dashed line) and the $\beta = 1.0$ and $r_s = 1 h^{-1}\text{Mpc}$ ReBEL model simulations 180B1RS1 (red dashed line). The thin vertical dashed line marks the Nyquist scale $R_{\text{Nyq}} \approx 1.4 h^{-1}\text{Mpc}$. The error-bars correspond to the 1σ scatter of the 180LCDM ensemble.

tio $S_4^{\text{ReBEL}}/S_4^{\Lambda\text{CDM}}$. In the panels we follow the same nomenclature and line scheme as in the previous section(s): the canonical 180LCDM ΛCDM model is indicated by the black solid line, the 180B-05RS1 ReBEL model with $\beta = -0.5$ and $r_s = 1 h^{-1}\text{Mpc}$ by the blue dotted line, the 180B02RS1 ReBEL model with $\beta = 0.2$ and $r_s = 1 h^{-1}\text{Mpc}$ by the green dot-dashed line and the 180B1RS1 ReBEL model with $\beta = 1.0$, $r_s = 1 h^{-1}\text{Mpc}$ by the red dashed line. Also, we indicate the Nyquist scale $R \sim 1.4 h^{-1}\text{Mpc}$ again by means of the vertical dashed line.

TABLE IV: Values of the deviations ΔS_3 and ΔS_4 of the skewness and kurtosis, at a scale of $R \sim 1.4 h^{-1}\text{Mpc}$, measured for the ReBEL models from those of the canonical ΛCDM model. For the definition of ΔS_3 and ΔS_4 see Eqn. (27). First column: redshift z . Three additional columns: values for S_3 (top table) and S_4 (bottom table) for three $180 h^{-1}\text{Mpc}$ ReBEL ensemble simulations, 180B-05RS1, 180B02RS1 and 180B1RS1.

ΔS_3			
z	180B-05RS1	180B02RS1	180B1RS1
0.0	0.126	-0.066	-0.247
0.5	0.295	-0.038	-0.245
1.0	0.270	-0.103	-0.328
2.0	0.220	-0.100	-0.305
5.0	0.020	-0.014	-0.120
ΔS_4			
z	180B-05RS1	180B02RS1	180B1RS1
0.0	0.216	-0.154	-0.484
0.5	0.698	-0.075	-0.471
1.0	0.560	-0.219	-0.603
2.0	1.050	-0.242	-0.550
5.0	0.600	-0.045	-0.288

1. Deviation Scale

Earlier, we had noted that at $z = 0$ the ratio of the hierarchical amplitudes $S_n(R)/S_n^{\Lambda\text{CDM}}$ in the various ReBEL models to that in the ΛCDM model is close to unity on large scales, scales considerably in excess of the ReBEL scale parameter r_s and in the order of the scale of transition between linear and nonlinear evolution. When assessing this ratio for skewness and kurtosis at other redshifts, we notice the same trend.

Interestingly, there is a slight but seemingly systematic shift in the scale at which the skewness and kurtosis ratios start to deviate significantly from unity. We observe that this scale gradually shifts towards larger scale as the evolution proceeds. When looking at the scale $R_{5\%}$ at which the skewness of the ReBEL models differs more than $\sim 5\%$ from the ΛCDM skewness, in the case of the $\beta = 1.0$ ReBEL model we find that at $z = 5 h^{-1}\text{Mpc}$ it is only $R \sim 6 h^{-1}\text{Mpc}$ while at $z = 2$ it has increased to $R \sim 10 h^{-1}\text{Mpc}$ (see fig. 13). The observed trend is

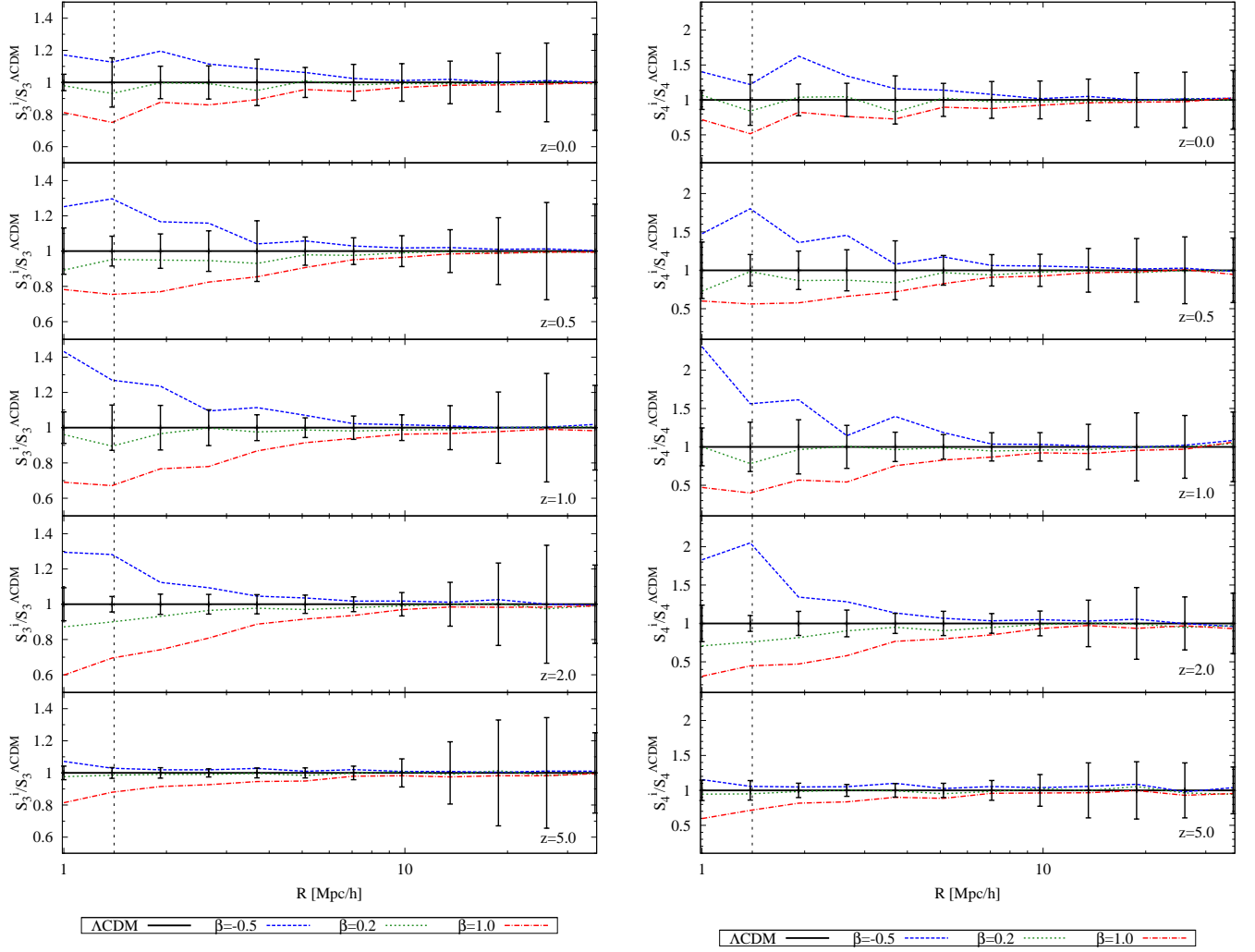


FIG. 11: (color on-line) Left column: redshift evolution of the skewness ratios $S_3^{ReBEL}/S_3^{\Lambda\text{CDM}}$ for different ReBEL models. The lines mark the ensembles 180LCDM (solid line), 180B-05RS1 (dashed line), 180B02RS1 (dotted line) and 180B1RS1 (dash-dotted line). Panels show redshifts for $z = 0$ (the top panel) to $z = 5$ (the bottom panel). Vertical dashed line marks the Nyquist scale $\sim 1.4 h^{-1}\text{Mpc}$. Righthand panel: identical set of redshift panels for the kurtosis ratio $S_4^{ReBEL}/S_4^{\Lambda\text{CDM}}$. The error-bars correspond to the 1σ scatter of the 180LCDM ensemble.

directly linked to the scales on which the density field reaches non-linearity: the hierarchical amplitudes can only start to deviate from the canonical ΛCDM values through the related strong mode couplings. The other ReBEL models display similar evolutionary trends, although the details may differ somewhat.

At more recent redshifts, in all ReBEL models the growth of the deviations slows down, and at $z = 0$ the scale is still $R_{5\%} \sim 10 h^{-1}\text{Mpc}$. Despite the growing amplitude of fluctuations at small nonlinear scales and the corresponding deviations of the ReBEL moments at these scales, the dynamical screening mechanism does not lead to the spread of these deviations to scales larger than $\sim 10 h^{-1}\text{Mpc}$. We may expect this, since the dynamical impact of the additional ReBEL scalar force will be rendered insignificant for Fourier modes smaller than

the comoving Fourier mode $k_s = 2\pi/r_s$ [17]. The required strong mode coupling will therefore not materialize. This observation is in agreement with the behaviour of the power spectrum $P(k)$ of the density perturbations, as noted in [18, 20]. Figure 13 illustrates the convergence of the deviation scale $R_{5\%}$ in the case of all three ReBEL models.

2. Redshift Dependence

Another interesting question with respect to the deviations of the ReBEL model skewness and kurtosis from the ΛCDM models concerns the issue at which redshift these are expected to be optimal. To address this issue, we assess the ReBEL S_3 and S_4 deviations on a scale of

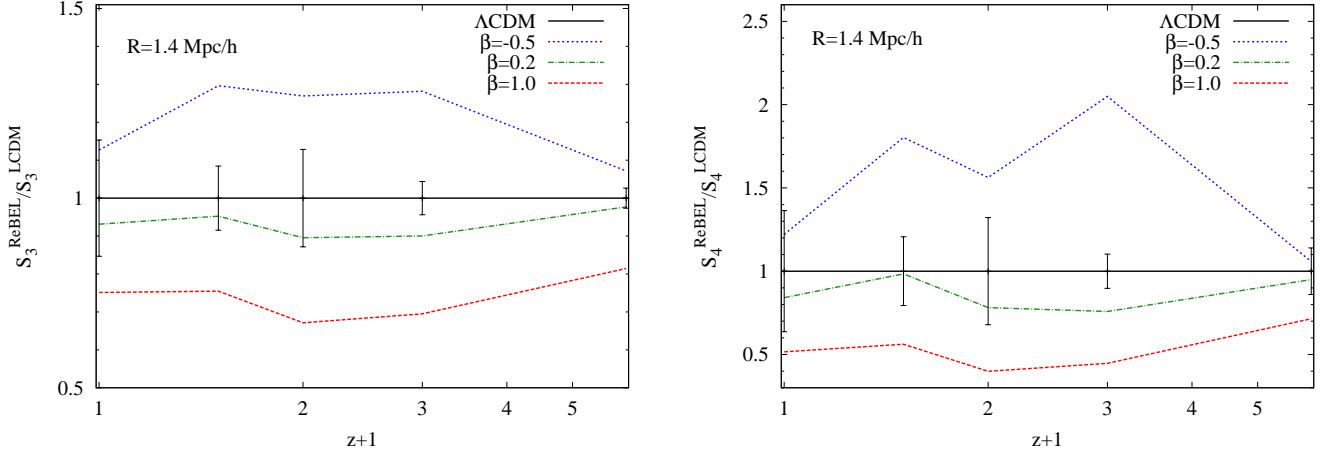


FIG. 12: (color on-line) Skewness ratio $S_3^{ReBEL}/S_3^{\Lambda CDM}$ (lefthand panel) and kurtosis ratio $S_4^{ReBEL}/S_4^{\Lambda CDM}$ (righthand panel) as a function of redshift z . Plotted are the ratios for three different ReBEL models: 180LCDM (solid line), 180B-05RS1 (dashed line), 180B02RS1 (dotted line) and 180B1RS1 (dotted-dashed line). The error-bars correspond to the 1σ scatter of the 180LCDM ensemble.

$\sim 1.4 h^{-1}$ Mpc scale. At this scale we find the highest deviations within the range set by the Nyquist scale.

In table IV we list the values of the skewness and kurtosis deviations ΔS_3 and ΔS_4 , determined for the $180 h^{-1}$ Mpc ReBEL simulation ensembles 180B-05RS1, 180B02RS1 and 180B1RS1. The magnitude of the hierarchy amplitude deviations ΔS_n is defined as:

$$\Delta S_n = \left(\frac{S_n^{ReBEL}}{S_n^{\Lambda CDM}} - 1 \right). \quad (27)$$

Interestingly, the most pronounced discrepancies between the ReBEL and the standard model DM skewness

and kurtosis are not found at the present epoch $z = 0$. Instead, we find the maximal deviations in the range $0.5 < z < 2.0$. This is directly confirmed by the visual inspection of the two panels in figure 12, where we plotted $1 + \Delta S_3$ and $1 + \Delta S_4$, i.e. the ratios $S_3^{ReBEL}/S_3^{\Lambda CDM}$ and $S_4^{ReBEL}/S_4^{\Lambda CDM}$, versus redshift z . Amongst the rather limited redshift archive at our disposal, the maximum appears to be found at $z = 1$. For a more precise determination of this epoch, we would need a considerably more densely binned redshift archive. Nonetheless, taking into account the errors in the amplitude estimates, we may confidently locate the maximum somewhere in the range quoted at the beginning of this paragraph.

In our numerical experiments, at $z = 1$ the ΔS_3 reaches 10.3% for the 180B02RS1 ensemble and 32.8% for the 180B1RS1 ensemble. The S_4 deviations are considerably larger, and attain values of 21.9% and 60.3% for the same ensembles. We should emphasize that even while the deviations appear to reach their maximum at around $z = 1$, they are still substantial at the current epoch, attaining values of 24.7% and 48.4% for the skewness and the kurtosis in the 180B1RS1 ReBEL ensemble.

We therefore reach the important conclusion that the sharpest “fingerprint” of the scalar dark matter interactions, in terms of skewness and kurtosis, should be found at moderately intermediate redshifts.

The answer to the question if these signatures can be detected in the observational reality depends on a range of issues. One of the most important ones is that of the bias between the dark matter distribution and the baryon density and galaxy distribution. As yet, there is not a definitive insight into how much this will be influenced by ReBEL dark matter forces. Keselman, Nusser and Peebles[20] recently studied the growth of cosmic structure in a simulation containing dark matter and baryons. While they confirm the expectation that the effect of

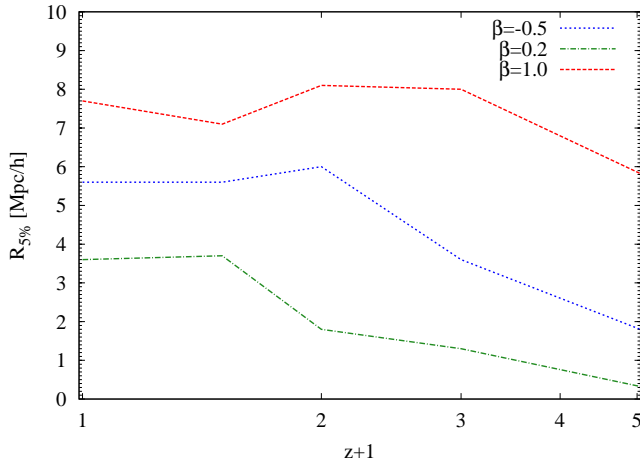


FIG. 13: (color on-line) Evolution of skewness deviation scale $R_{5\%}$ as a function of redshift. At the scale $R_{5\%}$ the skewness for the ReBEL model deviates by 5% from the value for the canonical ΛCDM model. Plotted are the deviation scales for three different ReBEL models: 180LCDM (solid line), 180B-05RS1 (dashed line), 180B02RS1 (dotted line) and 180B1RS1 (dotted-dashed line).

ReBEL forces on the small scale baryon distribution is much smaller than that on dark matter, they also find that cannot be ignored. Preliminary results of our high-resolution joint simulation of baryons and DM within the ReBEL model (work in preparation) shows that the impact of the scalar interactions is not only imprinted on the moments of the dark matter density field, but also on the baryon density field. The implication may be that it will indeed be feasible to put observational constraints on the ReBEL cosmology parameter space with the help of galaxy catalogues.

VII. DISCUSSION

In this paper we have addressed the question in how far the differences between cosmological models involving a scalar long-range dark matter interaction would distinguish themselves from the canonical Λ CDM models in terms of their statistical properties. In answering this question, we focussed on the hierarchy of correlation functions and moments of the cosmological density fields.

On the basis of a large ensemble of cosmological N-body simulations within the context of the standard Λ CDM cosmology and a range of different ReBEL long-range interaction models, we have attempted to identify the statistical differences between the models and the parameters and circumstances which will optimize our ability to discriminate between the different cosmologies. The simulations in this study are restricted to the pure dark matter distribution.

To measure the moment and correlation functions, we base ourselves on the cumulants of the counts in cells of the particle distributions produced by the various N-body simulations. In the first stage of our study, we have thoroughly checked the accuracy and reliability of our implementation of the Counts in Cells method. We also assessed the influence of practical limitations, such as that of the finite length of a simulation box, on the measurements of the moments. This is particularly crucial given the circumstance that the gravitational force in the ReBEL models includes an intrinsic scale length. The conclusion from our experiments is that our implementation of the CIC method successfully recovers the known results from perturbation theory in the context of the canonical Λ CDM model.

Subsequently, we have applied our toolbox to the measurement of higher order N-point correlation functions, from $N = 2$ to $N = 9$, and the related hierarchical amplitudes, from the skewness S_3 and kurtosis S_4 to S_8 . Amongst the most outstanding conclusions are:

- At scales comparable to the screening length parameter r_s of the ReBEL model, the N-point functions ξ_N and the hierarchical amplitudes S_n show deviations from the values expected in the standard Λ CDM cosmology.
- In general, the amplitudes S_N become smaller as

the DM force strength parameter β is larger. In the hypothetical situation of a negative β , the S_n are larger than in the case of the Λ CDM cosmology. Usually, the magnitude of the $\beta = 0.2$ model values S_n are still in the order of the Λ CDM values, which technically correspond to the $\beta = 0$ values.

- In a detailed comparison between the skewness and kurtosis of dark matter density fields, we find that the relative deviation of the kurtosis in the ReBEL models from that in the Λ CDM model is considerably larger than that for the skewness. In general, this is true for the whole range of hierarchical amplitudes: the deviation of S_n in the ReBEL models is larger when it concerns a higher order n .
- The deviations of the hierarchical amplitudes $S_n(R)$ in the ReBEL models from that in the Λ CDM model are larger at smaller scales R . At scales where the evolution of the density field is still in the linear or quasi-linear regime, the deviations are negligible. Only at highly nonlinear scales we notice substantial and measurable differences.
- The scale at which we find substantial differences between $S_n(R)$ in the ReBEL models and in the canonical Λ CDM model gradually grows in time, a direct manifestation of the increasing scale of non-linearity as a result of cosmological structure growth. However, this increase comes to a halt when nonlinear structure growth has proceeded towards scales where the intrinsic screening length of the ReBEL forces calls a halt to its impact on the dark matter distribution.
- The deviations of the skewness $S_3(z)$ and kurtosis $S_4(z)$ of the ReBEL model from those in the Λ CDM cosmology reach their maximum in the moderate redshift range $0.5 < z < 2$. In other words, the imprint of ReBEL forces in the N-point correlation functions should be expected to be more prominent at medium redshift than at the current epoch.

By confirming that there are noticeable differences in the higher-order clustering patterns between the standard Λ CDM cosmology and that in the ReBEL long-range dark matter interaction cosmologies we have identified a viable path towards constraining or falsifying these models on the basis of the observed galaxy distribution at moderate redshifts. Nonetheless, to be able to substantiate these claims we need to extend this analysis to more elaborate models. First, we need to assess whether the same significant conclusions may be drawn when the density field is sampled on the basis of dark matter halos and galaxies. This is a particularly important issue as the small measured differences between the Λ CDM and the ReBEL models might be washed out in the observationally relevant situation where the estimates are inferred from the dark matter halo distribution. Also, we need to

investigate the extent to which these findings are influenced by working in redshift space instead of in regular (comoving) physical space. We foresee a substantial impact of the short-range ReBEL forces.

In our upcoming study, we will address these questions on the basis of mock galaxy survey models, which will allow a direct comparison with circumstances prevailing in the observational reality.

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